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Dependence of Macrophysical Phenomena on the Values of the Fundamental Constants [and Discussion]

W. H. Press, A. P. Lightman, Rudolf Peierls and T. Gold

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Dependence of macrophysical phenomena on the values of the fundamental constants

By W. H. Press and A. P. Lightman
Harvard-Smithsonian Center for Astrophysics, Cambridge, Massachusetts 02138, *U.S.A.*

Using simple arguments, we consider how the fundamental constants determine the scales of various macroscopic phenomena, including the properties of solid matter; the distinction between rocks, asteroids, planets, and stars; the conditions on habitable planets; the length of the day and year; and the size and athletic ability of human beings. Most of the results, where testable, are accurate to within a couple of orders of magnitude.

1. Introduction

We have been given the pleasant task of reviewing the manner in which the fundamental constants affect our daily lives. Why are 'things' the way they are instead of some very different way? What sets the scale of the size of planets, the size of people, how fast we run, the length of the year, and so on?

This general area of diversionary physics has attracted the occasional attention of researchers for quite a while. Galileo considered scaling limits on living creatures, based on strength of materials arguments. More recently, we can trace a lineage including works by Haldane (1928), Weisskopf (1969, 1975), Salpeter (1966), Carter (1974), Carr & Rees (1979), and others. We will draw freely on this material in the present review.

The general organization of this paper will become clear to the reader as he or she proceeds. Roughly, the material is presented in order of increasing speculation.

2. Properties of solid material

(*a*) *Atomic properties*

Solid material has a dull and straightforward existence. It occupies a régime where only two characteristic masses are important, the mass of the proton m_p and the mass of the electron m_e. The only two forces of any importance are the electrostatic force of attraction, which brings in the electronic charge e and the degeneracy force or 'Pauli exclusion-principle force', which brings in Planck's constant $\hbar$. Relativistic effects are not important in this régime, so the speed of light c plays no role. Likewise, gravitational effects are not important, so Newton's constant G plays no role.

The basic scale of atomic distances is the Bohr radius a_0, where electric and degeneracy forces balance,

$$a_0 = \hbar^2/m_e e^2 \approx 0.53\times 10^{-8}\ \text{cm}, \tag{1}$$

and the basic scale of the energy in atomic processes is the binding energy of two charges separated by a Bohr radius, the rydberg

$$Ry = e^2/2a_0 \approx 2.2\times 10^{-11}\ \text{erg}\dagger = e^4 m_e/\hbar^2 = \alpha^2 m_e c^2. \tag{2}$$

† 1 erg = 10^{-7} J.

The final equality follows from the definition of the electromagnetic fine structure constant $\alpha = e^2/\hbar c \approx 1/137$, and shows why atomic processes are non-relativistic.

The characteristic frequency of atomic electronic transitions is

$$\omega_0 = Ry/\hbar \sim e^4 m_e/\hbar^3 \approx 4\times 10^{16}\ \text{s}^{-1}.\dagger \quad (3)$$

Equivalently, the wavelength of light emitted in atomic processes, $2\pi c/\omega_0$, is about $2\pi \times 137$ Bohr radii, which is somewhat shorter than 0.1 μm and is what we call ultraviolet.

The characteristic density of all solid matter is of the order of one proton mass per cubic Bohr diameter,

$$\rho_0 = m_p/(2a_0)^3 \approx 1.4\ \text{g cm}^{-3}. \quad (4)$$

The characteristic bulk modulus (compressibility) of all solid matter is of the order of its internal binding energy density, namely

$$\mu_b \sim Ry/(2a_0)^3 \approx 1.8\times 10^{13}\ \text{dyn cm}^{-2}\ddagger. \quad (5)$$

The characteristic coefficient of thermal expansion is

$$k/Ry \approx 6.3\times 10^{-6}\ \text{K}^{-1}. \quad (6)$$

Here k is Boltzman's constant, which we can usually avoid by writing all temperatures in units of energy (e.g. 1 eV $\approx 10^4$ K).

In order of magnitude, the opacity of matter to electromagnetic radiation is set by the Thomson cross section

$$\sigma_T = \tfrac{8}{3}\pi\,(e^2/m_e c^2)^2. \quad (7)$$

Measured as a total cross section per bulk mass, the opacity is

$$\kappa_T = \sigma_T/m_p \approx 0.4\ \text{cm}^2\ \text{g}^{-1}. \quad (8)$$

While opacities near to atomic resonances may be much larger than this, and much smaller far away from resonances when all electrons are tightly bound, equation (8) is generally the only dimensional scale that can enter.

(b) *Molecular properties*

The characteristic vibrational frequencies of many molecular bonds are smaller than those of electronic transitions by a factor $(m_e/m_p)^{\frac{1}{2}}$, basically because the entire mass of the molecule, not just the electronic cloud, participates in such modes. Correspondingly, the characteristic bond energies are down by the same factor from the rydberg. In typical solid crud (we use 'crud' in its technical sense, meaning a substance which is neither metallic nor single-crystalline), these weaker bonds are responsible for shear rigidity, and for holding the crud together under forces of extension. Therefore, the typical shear modulus or tensile strength of crud is set in scale by

$$\mu_s \sim [Ry/(2a_0)^3]\,(m_e/m_p)^{\frac{1}{2}} \approx 4\times 10^{11}\ \text{dyn cm}^{-2}. \quad (9)$$

In practical units, this is about 6×10^6 lb/in².§ Real materials always contain structural flaws and are an additional factor of 10 or 100 below this.

† We use the symbol ~ to mean 'of order'.
‡ 1 dyn = 10^{-5} N.
§ 1 lb ≈ 0.453 kg, 1 in = 2.54×10^{-2} m.

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The characteristic thermal conductivity of solid crud is set by its lattice spacing and lattice vibrational frequency. As for all transport phenomena, the transport coefficient is the product of a number density of carriers (here of order a_0^{-3}), a mean free path or phonon wavelength (here of order a_0), a transport velocity (a_0 times equation (3) times $(m_e/m_p)^{\frac{1}{2}}$ divided by $\hbar$), and a specific heat per molecule k. The result is

$$(\text{conductivity}) \sim (Ry/a_0\hbar)\,(m_e/m_p)^{\frac{1}{2}}k \approx 1\times 10^7 \text{ erg cm}^{-1}\text{ K}^{-1}\text{ s}^{-1}. \tag{10}$$

In fact, like (9), this estimate is about a factor of a hundred too large, since real materials are full of dislocations and other phonon-scattering unpleasantness.

For what follows below, we will need to take cognizance of the fact that an important set of complex chemical phenomena take place with bond energies that are even another factor of ten smaller than the bond energies in crud. This extra factor of about 0.1, which we will denote by the symbol ϵ, does not arise out of any combination of physical constants, but comes from all the abhorrent details of chemistry that are omitted in this paper. Substances that are made of these delicate bonds are called 'organic'. The characteristic bond energy of organic substances is then of order

$$\epsilon Ry(m_e/m_p)^{\frac{1}{2}}. \tag{11}$$

The temperature corresponding to this bond energy, i.e. such that kT is of order expression (11), is about 350 K. More on this in §4 below.

3. Self-gravitating objects

(a) *Energy scalings*

Gravity is so weak a force as to be negligible except when very large numbers of atoms are aggregated together. Let us consider an aggregation of N atoms in a configuration such that their mean separation is some distance d. Then the aggregate mass and size of the resulting object are

$$M \sim Nm_p \qquad R \sim N^{\frac{1}{3}}d. \tag{12}$$

The gravitational binding energy (g.e.) of the object is

$$\text{g.e.} \sim GM^2/R \sim G(Nm_p)^2/N^{\frac{1}{3}}d. \tag{13}$$

The degeneracy energy (d.e.) of the object is obtained by application of the uncertainty principal to get a Fermi momentum ($p_F d \sim \hbar$), and converting this to a Fermi energy by $E_F \sim p_e^2/(2m_e)$, giving

$$\text{d.e.} \sim N(\hbar/d)^2/2m_e. \tag{14}$$

If the object is at a finite temperature T, then it has a total thermal energy (t.e.) content of

$$\text{t.e.} \sim NkT, \tag{15}$$

and a total radiation energy (r.e.) given by the product of its volume with aT^4, where a is the radiation constant:

$$\text{r.e.} = Nd^3aT^4, \quad a \equiv \tfrac{1}{15}\pi^2\, k^4/c^3\hbar^3. \tag{16}$$

The object's electrons, which are the lightest particles and thus the first to become relativistic as temperature or density is increased, have a total rest-mass energy (e.m.e.)

$$\text{e.m.e.} \sim Nm_e c^2. \tag{17}$$

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Finally, the total rest mass energy of the object is

$$\text{p.m.e.} \sim Mc^2 \sim Nm_p c^2. \tag{18}$$

We give the various classes of objects in the Universe different names according to which of the above forms of their energy dominate, which balance, which are negligible compared with the others, and which (if any) are comparable with the chemical or molecular binding energy densities of (5) or (9). We use such names as rocks, asteroids, planets, stars, black holes, and so forth.

(*b*) *Rocks, asteroids, planets*

Consider the classes of objects for which the degeneracy energy equation (14) dominates *all* the others when d has the value of the Bohr radius a_0. Then the degeneracy energy gives the object a bulk incompressibility given numerically by (5). As we saw in §2, such objects will consist of ordinary matter ('crud'), at ordinary density given by (4). We call such objects 'rocks' or, as they get larger, 'asteroids'. Gravitation is entirely unimportant for these objects.

There is no minimum size to a rock. There is, however, a maximum size to an asteroid, defined to be the size where gravitational forces are able to overcome the shear modulus of the material and thus cause it to become spherical instead of arbitrarily rock-shaped. We can calculate this size in order of magnitude by equating the gravitational energy (13) to the volume times the shear modulus (9), and using the relations (12). This gives a mass of

$$M \sim \tfrac{1}{64}(e^2/Gm_p^2)^{\frac{3}{2}}\,(m_e/m_p)^{\frac{3}{4}}\,m_p \approx 1\times10^{26}\,\text{g}. \tag{19}$$

This mass is the maximum mass of an asteroid or, equivalently, the minimum mass of a planet, a planet being defined as an object made of crud that is rendered spherical by gravitational forces. Equation (19) gives a result that is a rather larger than the actual number, because the actual shear strength of planetary matter is less than its dimensional value; that is geology, not physics, but we will need to say more about it below.

It is convenient to define the gravitational fine structure constant $\alpha_G \equiv Gm_p^2/\hbar c \approx 5.88\times10^{-39}$, so that one can write the first dimensionless combination on the right side of (19) as $(\alpha/\alpha_G)^{\frac{3}{2}}$. Then, rewriting (19), the minimum mass and radius of a planet are, roughly,

$$M \sim m_p(\alpha/\alpha_G)^{\frac{3}{2}}\,(m_e/m_p)^{\frac{3}{4}}, \quad R \sim a_0(\alpha/\alpha_G)^{\frac{1}{2}}\,(m_e/m_p)^{\frac{1}{4}}. \tag{20}$$

The maximum mass of objects that we call planets is set by the condition that the gravitational energy does not dominate the degeneracy energy. When mass is increased so that the gravitational energy *does* dominate the degeneracy energy at normal density, then the object is called a 'cold, degenerate-dwarf star'. The dividing mass is given by equating (13) and (14), with $d = a_0$. This gives

$$M \sim (\alpha/\alpha_G)^{\frac{3}{2}}\,m_p \approx 2\times10^{30}\,\text{g}. \tag{21}$$

Comparing with (20), we see that planets exist for a mass range of only about $(m_e/m_p)^{-\frac{3}{4}} \approx 300$. The low end of this range is somewhat below the mass of the Earth, while the high end of the range is at about the mass of Jupiter. (Jupiter is almost a stillborn star.)

(c) *Cold, degenerate stars*

Cold, degenerate stars range upwards in mass from the value given by (21), corresponding to d decreasing below a_0 and densities increasing above those of normal matter. We know very little about how many such objects there are in the Universe. At the low end of their mass range these objects will never have ignited their nuclear fuel, so they are dark and virtually unobservable. They may be the 'missing mass' that is observed gravitationally in galaxy rotation curves; that subject is one of current debate in astronomy.

There is an upper limit to the possible mass of cold, degenerate stars, first found by Chandrasekhar (1931 *a*, *b*). The relation between mass and radius for such stars is obtained by equating (13) (gravitational binding energy) and (14) (degeneracy energy). One finds that the radius varies as the $-\frac{1}{3}$ power of mass. Along this sequence, the other forms of energy, equations (15)–(17), are negligible (in part by definition, since $T \equiv 0$). As the mass approaches some maximum value M_C, however, the degeneracy energy approaches the rest energy of the electrons, equation (17). When the electrons become relativistic, it can be shown (Landau 1932) that the solution of the equation of hydrostatic equilibrium gives an unstable, rather than a stable, solution.

This maximum mass M_C of cold, degenerate stars is thus obtained by equating (13), (14), and (17), eliminating d and solving for N. The result is the *Chandrasekhar mass*

$$M_C \sim (\hbar c/Gm_p^2)^{\frac{3}{2}}\, m_p = \alpha_G^{-\frac{3}{2}}\, m_p \approx 3.7\times 10^{33}\ \mathrm{g}. \tag{22}$$

Notice that the Chandrasekhar mass is a factor of $(137)^{\frac{3}{2}} \approx 1600$ larger than (21), so this is the mass range over which cold, degenerate stars are possible.

As we will now see, aggregations of matter at the upper end of this range *are* capable of igniting their nuclear fuel. In the real Universe, therefore, cold, degenerate dwarfs in the upper part of the range occur as the cinders of formerly luminous stars and not as primordial objects.

(d) *Luminous stars*

A luminous star, as opposed to a cold degenerate star, is one in which thermal energy dominates degeneracy energy. Gravitational force is balanced with thermal pressure, so the gravitational energy (13) and thermal energy (15) must be about equal. That gives one relation between the mass M, radius R, and average (or central) temperature T of the star. We need one further relation before we can solve for the radius and temperature of stars as a function of their mass.

The extra relation is an equation that relates the central temperature of a star to the *logarithm* of fundamental constants, R, and M. The logarithm is so slowly varying that it is, for all order-of-magnitude purposes, a constant. (One can solve for the self-consistent values of R and M that go into the logarithm by iteration, starting with virtually any first guesses.)

The key idea is this: we call something a star if the time that it takes to burn all its nuclear fuel is longer than its thermal diffusion time. This is the condition for it to be able to settle down to a radiative equilibrium. Something that releases nuclear energy faster than its thermal diffusion time is a nuclear explosion, not a star.

The thermal diffusion time is the time that it takes a photon to random-walk its way through the distance R. This equals the light travel time across that distance times the optical depth. The optical depth is related to M, R, and the opacity (8) by

$$\tau_{\mathrm{opt}} \sim M\sigma_T/R^2 m_p, \tag{23}$$

so the radiative diffusion time (also called the Kelvin–Helmholtz time) is

$$t_{KH} \sim M\sigma_T/Rcm_p \approx 1\times10^4 \text{ a}, \tag{24}$$

where the numerical value is calculated using $M = 2\times10^{33}$ g, $R = 7\times10^{10}$ cm, the actual values for the Sun. The above timescale is actually about 100 times too small, due to our underestimate of the opacity by that factor.

The rate of nuclear burning, that is the inverse lifetime of a typical particle against a nuclear reaction, is given by the familiar expression involving density, particle velocity, and cross section σ_{nuc}

$$\lambda_{nuc} \sim (M/R^3)\,\sigma_{nuc} v/m_p, \tag{25}$$

where v is a typical thermal velocity of order $(kT/m_p)^{\frac{1}{2}}$. (We should note here that detailed stellar models are centrally condensed and have central densities that are about 100 times larger than M/R^3.)

The nuclear cross section is that appropriate for weak interactions with a Coulomb barrier (e.g. $p+p\rightarrow p+n+e^+ +\nu$). Some detailed nuclear kinematics shows that this varies with energy E as (see, for example, Clayton 1968)

$$\sigma_{nuc} = (S/E)\exp(-2\pi e^2/\hbar v), \tag{26}$$

where S is the so-called 'astrophysical cross section factor'. The calculation of S from first principles will bring in the weak interaction constant $G_F \sim \alpha/m_W^2$, where m_W is the mass of the intermediate boson, as well as various phase space factors involving m_p and m_e. Since, however, we will only need the logarithm of S, we will not try to track through these details, but rather use its measured value, which is on the order of 10^{-21} keV b.†

We now get the central temperature of a luminous star by setting the product of (25) and (24) equal to unity and then taking the logarithm. This gives, collecting factors together,

$$\frac{2\pi e^2}{\hbar(kT/m_p)^{\frac{1}{2}}} \sim \ln\left[10^8\times\left(\frac{M}{m_p}\right)^2\left(\frac{\sigma_T}{R^2}\right)\left(\frac{kT}{m_p c^2}\right)^{\frac{1}{2}}\left(\frac{S}{R^2 kT}\right)\right]. \tag{27}$$

The factor 10^8 has been inserted into the logarithm to compensate for the underestimates of t_{KH} and λ_{nuc} which were already mentioned, and also to reflect the fact that most objects that we call stars live much *longer* than one Kelvin–Helmholtz time. In part, this is observational selection: we see the relatively long-lived objects. For 'correct' values of R, M, and T, and S, the logarithm has the value of about 15, which can be found self-consistently once M and R are determined.

Rewriting (27), we obtain the central temperature of all stars,

$$kT_* \approx m_p c^2(2\pi\alpha^2)\left(\tfrac{1}{15}\right)^2 \sim 0.1\alpha^2 m_p c^2 \approx 10 \text{ keV}. \tag{28}$$

Actual central temperatures are about a factor 10 smaller than this, due in part to the fact that most reactions occur several e-folds out on the tail of the Maxwellian thermal distribution. The above derivation is essentially that of Salpeter (1966). Wherever the explicit factor 0.1 appears below, it is to be traced back to the logarithmic numerical factor in (28). We now can equate (13) and (15) and obtain the linear relation between the mass and the radius of luminous (main-sequence) stars,

$$M/R \sim 0.1c^2\alpha^2/G \approx 7\times10^{22} \text{ g cm}^{-1}. \tag{29}$$

† 1 b = 10^{-28} m².

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Since radiation density, which is the conserved quantity diffused through the star, goes as T^4, the surface temperature T_s of a star of mass M is down from its central temperature by a factor of $\tau_{opt}^{\frac{1}{4}}$, where τ_{opt} is given by (23). So,

$$T_s \sim \frac{T*}{\tau_{opt}^{\frac{1}{4}}} \sim 0.1 \frac{m_p c^2}{k} \alpha^2 \left(\frac{R^2 m_p}{M^2 \sigma_T}\right)^{\frac{1}{4}} M^{\frac{1}{4}}$$

$$\sim \sqrt{(0.1)}\, \alpha \frac{m_p c^2}{k} \left(\frac{\alpha_G}{\alpha}\right)^{\frac{1}{2}} \left(\frac{m_e}{m_p}\right)^{\frac{1}{2}} \left(\frac{M}{m_p}\right)^{\frac{1}{4}} \approx 10^5 \text{ K}, \tag{30}$$

where (29) has been used in going from the first line to the second. Actual stars have surface temperatures about a factor ten less than this estimate, primarily due to more complicated effects in the opacity.

The luminosity of a star of mass M follows from (30), (29), and from the radiation law,

$$L \sim acT_s^4 R^2 \sim (c^5/G)\, \alpha_G^5 \alpha^{-2} (m_e/m_p)^2\, (M/m_p)^3. \tag{31}$$

Equation (31) is an unlucky case where a slightly more detailed treatment (Lightman 1983) shows that neglected factors of order unity have built up to a significant factor. The factor $\pi/(3^3 \times 2^5 \times 15) \approx 2 \times 10^{-4}$ should be inserted in front of the right side. With this factor, (31) gives numerically (for the Sun's mass) about 6×10^{33} erg s^{-1}, as compared with the actual value of about 4×10^{33} erg s^{-1}. It is interesting to note that, although the equations which led to (31) contained the assumed central temperature (28), that temperature actually cancels out of (31).

So we now know how the properties of stars vary with their mass. But what are the upper and lower limits to the masses of stars along this natural sequence?

The lower limit is set by the requirement that a luminous star that satisfies the mass–radius relation (29) have a degeneracy energy (14) that is smaller than its gravitational or thermal energy content. If this were not so, then the star would never have condensed down to the point of igniting its nuclear fuel: it would have come to equilibrium as a cold, degenerate star at a larger radius than that required by (29). Manipulating (14), (12), and (29), we can write this minimum mass as

$$M \sim (0.1 m_p/m_e)^{\frac{3}{4}}\, (\alpha/\alpha_G)^{\frac{3}{2}}\, m_p \approx 1 \times 10^{32} \text{ g}, \tag{32}$$

which is larger than the maximum mass of a planet or minimum mass of a cold, degenerate star (21) by the factor $(0.1 \times 1836)^{\frac{3}{4}} \approx 50$.

The maximum mass of a star is set by the requirement that the total energy of thermal radiation (16) should not dominate the thermal energy (15). If this becomes violated, then the star, now pressure-supported by a relativistic photon gas, becomes hydrostatically unstable, much as the degenerate stars did when their electrons became relativistic. The limiting mass is thus obtained by equating

$$(M/m_p)\, kT \sim GM^2/R \sim R^3 a T^4. \tag{33}$$

If T is eliminated between these equations, one finds miraculously that R also cancels out, giving a mass limit independent of our assumed central temperature,

$$M \sim \alpha_G^{-\frac{3}{2}}\, m_p \approx 4 \times 10^{33} \text{ g}. \tag{34}$$

This is dimensionally just the same as the Chandrasekhar mass limit for cold, degenerate stars, although the physical situation contemplated is quite different. Because of neglected factors of order unity in (34), one is actually able to find stars up to about ten times the indicated mass.

Comparing (34) with (32), we see that luminous stars are possible over a mass range of only about $137^{\frac{3}{2}}/50 \approx 30$, ranging down from somewhat larger than the Chandrasekhar mass.

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4. Habitable planets

We are now in a position to say something not just about planets in general, but about planets with an environment conducive to the evolution of complicated organic structures ('organic' having been defined by (11)).

(a) *Mass and radius of the Earth*

Since the Earth is made of ordinary crud, its mass M and radius R lie on the ordinary density relation (4),

$$M/R^3 \sim m_p/(2a_0)^3. \tag{35}$$

We need an additional relation that distinguishes planets in general from habitable planets. That relation follows from positing that a habitable planet must have a differentiated atmosphere (neither vacuum nor primordial hydrogen and helium) that is gravitationally bound to the planet at the temperature of expression (11), where interesting organic reactions can take place. Therefore, the escape velocity from the surface of a habitable planet should be greater, but not too much greater, than the thermal velocity of hydrogen at that temperature (Press 1980). This condition gives

$$GM/R \sim \epsilon (m_e/m_p)^{\frac{1}{2}}\, Ry/m_p. \tag{36}$$

Now, (36) and (35) can be solved for M and R separately, giving

$$\left.\begin{aligned} R &\sim \epsilon^{\frac{1}{2}}(2a_0)\,(m_e/m_p)^{\frac{1}{4}}\,(\alpha/\alpha_G)^{\frac{1}{2}} \approx 5\times 10^8\ \text{cm},\\ M &\sim \epsilon^{\frac{3}{2}} m_p\,(m_e/m_p)^{\frac{3}{4}}\,(\alpha/\alpha_G)^{\frac{3}{2}} \approx 2.5\times 10^{26}\ \text{g}, \end{aligned}\right\} \tag{37}$$

where the numerical values are evaluated using $\epsilon = 0.1$ (see discussion preceeding expression (11)). These values are in moderately good agreement with the true values for the Earth, $R \approx 6.4\times 10^8$ cm and $M \approx 5.9\times 10^{27}$ g. Notice that the mass of a habitable planet is smaller than the maximum planetary mass (equation (21)) by a factor $\epsilon^{\frac{3}{2}}(m_e/m_p)^{\frac{3}{4}}$.

(b) *Length of the day and of the year*

All planets in the solar system, unless they are tidally locked to some other planet (Venus), a nearby moon (Pluto), or the Sun (Mercury), are observed to rotate at an angular velocity that is within an order of magnitude of centrifugal breakup. While the evolutionary details that lead to this circumstance are not well understood, the phenomenon seems likely to be universal. There is, in fact, an abundance of angular momentum in observed protostellar gas clouds. The angular velocity of centrifugal breakup depends only on the density, not on the mass and radius separately, and is of order $G\rho^{\frac{1}{2}}$. The length of a 'universal day' therefore follows from equation (4) (Lightman 1983),

$$T_{\text{day}} \sim \frac{2\pi}{G\rho_0^{\frac{1}{2}}} \sim 2\pi(2)^{\frac{3}{2}}\,\frac{a_0}{c}\left(\frac{m_p}{m_e}\right)^{\frac{1}{2}} \alpha^{-\frac{1}{2}}\,\alpha_G^{-\frac{1}{2}} \approx 6\ \text{h}. \tag{38}$$

The length of the year is not, of course, universal for all planets. However, its value is determined for habitable planets, as we have defined them: from (31), we know the luminosity of a star as a function of its mass. Equations (32) and (34) bracket the possible mass range of luminous stars into a narrow range. Let us take the geometric mean of these expressions as being characteristic of the 'typical' star, $M \sim 0.1 m_p \alpha_G^{-\frac{3}{2}}$. Now we can compute the distance d

of a habitable planet from its star by the requirement that it be in thermal equilibrium at the habitable temperature of (11). This, together with the black body law, and the above typical mass of a star, gives a determination of the 'astronomical unit' d in terms of fundamental constants

$$d \sim \left(\frac{60}{\pi^3 10^3}\right)^{\frac{1}{2}} a_0 \left(\frac{m_p}{m_e}\right) \epsilon^{-2}\alpha^{-4}\alpha_G^{-\frac{1}{4}} \approx 5\times 10^{13}\ \text{cm}. \tag{39}$$

When the above expressions for M and d are combined with Kepler's law, we obtain an expression for the year on a habitable planet in terms of fundamental constants (Lightman 1983)

$$T_{\text{year}} \sim 0.2(a_0/c)\ (m_p/m_e)^2\ \epsilon^{-3}\alpha^{-\frac{13}{2}}\alpha_G^{-\frac{1}{8}} \approx 20\ \text{a}. \tag{40}$$

All biological phenomena associated with the changing seasons may thus be expected to follow the universal time scale given by (40).

(c) *Height of mountains on earth*

Weisskopf (1975) has treated this topic beautifully: the pressure at the base of a mountain of height H is the product of its height, its density, and the acceleration of gravity. We equate this to the shear strength, (9), giving

$$H(GM/R^2)\ \rho_0 \sim [Ry/(2a_0)^3]\ (m_e/m_p)^{\frac{1}{2}}. \tag{41}$$

With equation (4) for universal density, both for the mountain and for the planet, (41) can be rewritten as the ratio of the mountain height to the planetary radius,

$$H/R \sim (\alpha/\alpha_G)\ (m_e/m_p)\ (m_p/M)^{\frac{2}{3}}. \tag{42}$$

Now, something strange turns out if we substitute into (42) expression (37) for the mass of a habitable planet. We get an answer greater than one! Mountains can apparently be larger than the Earth! This is a case where we have once again run afoul of a build-up of dimensionless numbers that do not involve the fundamental constants. In fact, observationally, (37) is about right, while (42) is a big overestimate. Weisskopf does a more careful job than we are doing here, and shows that a factor $A^{-\frac{5}{3}}$ appears on the right of (42), where A is the mean molecular weight of the material (around 50 for both Fe and SiO_2); Weisskopf also tracks through additional materials-related factors amounting to an additional factor of about 0.03. With these inserted, (42) is not too bad.

In fact, we could have pointed out the problem when we first derived (37) above, since that mass of a habitable planet is smaller than the minimum mass of a planet (20), by a factor $\epsilon^{\frac{3}{2}}$. Equation (20) is an overestimate. On geological timescales, matter is softer than the fundamental constants indicate at first glance, and by a significant factor. (One can imagine a more thorough treatment, where the rate of plastic flow of rock at habitable temperature is derived from fundamental constants, and is taken to be in equilibrium with the age of the earth. On *very* long time scales, as Dyson (1979) has pointed out, all solid objects become spherical.)

(d) *Sunlight and weather*

We have already seen that the solar flux at Earth is determined by the condition that the Earth be in approximate black-body equilibrium at habitable temperatures. There is also something else remarkable about sunlight: its spectral temperature (that is, the surface temperature of the Sun) is smaller than, but on the order of, a rydberg. Life would be very difficult

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were this not the case. If the spectral temperature of sunlight were much higher, then organic structures could not withstand its flux; it would effectively sterilize our planet. If the spectral temperature were much lower, then sunlight could not easily drive photosynthetic reactions (see, for example, Wald 1959).

In part, the happy spectral coincidence is not a coincidence at all, but is due to details of the opacity of solar material which we have neglected. Opacities get large at a fraction of a rydberg, *because* there is a lot of chemistry there, and we see into the Sun just to the point where the opacities get large, by definition of optical depth. However, if stellar surface temperatures estimated by (30) were very much larger, then the opacity would not have the opportunity to get large anywhere near the stellar surface. A numerical coincidence of the fundamental constants is involved: dividing (30) by (2) gives, after some simplification,

$$kT_s/Ry \sim \sqrt{(0.1)}\,(m_p/m_e)^{\frac{1}{2}}\,\alpha_G^{\frac{1}{8}}\,\alpha^{-\frac{3}{2}} \approx 0.3. \tag{43}$$

There is presumably no fundamental reason that this combination of constants had to come out to be of order unity.

So now the Sun is shining, and we can turn to the weather. The temperature of the atmosphere, as we saw, is by definition the habitable temperature $\epsilon(m_e/m_p)^{\frac{1}{2}}\,Ry/k$. What is the pressure of the atmosphere at the Earth's surface? We do not know of a believable estimate based on fundamental considerations. Comparing the Earth with Venus and Mars, whose atmospheric pressures are respectively of order $10^{\pm 2}$ times that of Earth, we see that the range of pressure can be large. It may be that the only answer is an anthropic one: our atmospheric column density is not too different from the reciprocal of (8), i.e. that density which provides a significant, but not overwhelming, opacity. Some opacity, we might argue, may be necessary to screen ultraviolet and cosmic radiation, while too much would produce a surface environment with no source of spectral free energy, just a local thermodynamic equilibrium at the habitable temperature. We would not wish to lean too heavily on these arguments.

It is somewhat easier to see what sets the scale of velocity of large-scale planetary winds on the Earth. The speed of sound in air follows from the habitable temperature,

$$v_s \sim (kT/m_p)^{\frac{1}{2}} \sim \epsilon^{\frac{1}{2}}\alpha(m_e/m_p)^{\frac{3}{4}}\,c \approx 2\times 10^5\ \mathrm{cm\ s^{-1}}. \tag{44}$$

It is not a coincidence that this value is about the same as the Earth's rotational velocity. That just restates the facts that the Earth rotates not too far from breakup velocity (equation (38)), and that the atmosphere is not too gravitationally bound at habitable temperature (equation (36)). By simple geometry, the black-body equilibrium temperature at extreme latitudes should be smaller than that at the equator by of order $2^{\frac{1}{4}}$ (the fourth-root coming from the usual radiation law). This differential, if not compensated by wind motion, would generate pressure differences of the same fractional amount, and cause pressure gradient driven winds on the order of a like fraction of the speed of sound. But since the rotational speed of the Earth is also of order the speed of sound, the geostrophic (circular) flow which balances those pressure gradients has comparable speed, and has a scale on the order of the size of the planet. So, winds must be of order some modest fraction of the sound speed. As we know from observation, that fraction is on the order of a few tenths (on planetary scales), scaling down by some Kolmogorov-like (but not fully three-dimensional) cascade on smaller scales. So, the dependence of wind velocity on the fundamental constants is the same as in (44).

5. The human condition in terms of fundamental constants

(a) *How big are we?*

While we might wish to define ourselves as humans according to our ability to reason, the physics of the matter is more prosaic: we are as highly evolved as we are because we are about as big as we can be without breaking when we fall. This criterion may not distinguish us from horses and elephants; as far as the fundamental constants know, they are human too. We can compute the implied human size h and mass $m \sim h^3\rho_0$ by equating our potential energy in a gravitational field of acceleration g to our breaking energy, the total bond energy of a 2-dimensional surface containing $(m/m_p)^{\frac{2}{3}}$ organic bonds (Press 1980):

$$mgh \sim \epsilon(m_e/m_p)^{\frac{1}{2}}\, Ry(m/m_p)^{\frac{2}{3}}. \tag{45}$$

Substituting the mass and radius of the Earth (equation (37)) into g gives, after some rearrangement,

$$h \sim \epsilon^{\frac{1}{4}}(m_e/m_p)^{\frac{1}{4}}\,(2a_0)\,(\alpha/\alpha_G)^{\frac{1}{4}} \approx 3\text{ cm}. \tag{46}$$

This number is observationally a factor of about 10^2 too small. The reason is that, rather as mountains were considerably softer than their dimensional strength, so we are considerably tougher. We are composed of a polymeric molecular structure that distributes shocks over a larger region than the weakest fault-plane of brittle, semi-crystalline crud.

Neglecting the smaller numerical factors, one sees that the human size (46) lies at the geometric mean of the planetary size (equations (20) or (37)) and the atomic size a_0 (Carr & Rees 1979). Also noteworthy is that the right side of (46) is also obtained when one computes either (i) the size of first-destabilized water waves on a lake (Weisskopf 1975), or (ii) the maximum size of water drops dripping off a ceiling. Both of those calculations involve a somewhat different physical situation, equation of surface tension to gravitational stress, but the dimensional combinations that result are the same. In a loose sense, this is why there is something special about the centimetre–metre scale on Earth, why the range of phenomenology is so rich on that scale: both solid and liquid 'things' have material strengths on about that scale.

(b) *How hard can we work?*

As already mentioned, there is no fundamental distinction between humans and horses. It is of interest, therefore, to compute the horsepower in terms of fundamental quantities.

Over a very large range of animal sizes, the peak power output of animals is observed to scale as a power not too different from the $\frac{2}{3}$ power of their mass (Wilkie 1959). Likewise, the resting metabolism of animals scales with mass with the same exponent. These data strongly suggest that the power output is limited by cooling through the animal's surface area, and that resting metabolism is scaled to keep the resting animal sufficiently warm.

In this case, we can compute the horsepower in terms of a tolerable thermal differential ΔT,

$$(\text{horsepower}) \sim \Delta T \times (\text{conductivity}) \times (\text{area})/(\text{skin depth}). \tag{47}$$

If we put in observed values 10 °C, 4×10^4 erg °C^{-1} cm^{-1} s^{-1}, 1 m^2, and 1 cm, (47) gives about 400 W, which is close to the observed value.

If we had no knowledge of the observed parameters, we could use $\Delta T \approx T$, area of order h^2, skin depth of order h (where h is given by equation (46)), and conductivity as given by (10). These values are

$$\text{(horsepower)} \sim \epsilon^{\frac{5}{4}} (Ry)^2/\hbar \, (m_e/m_p)^{\frac{5}{4}} \, (\alpha/\alpha_G)^{\frac{1}{4}} \approx 200 \text{ W}. \tag{48}$$

By equating skin thickness to h, rather than to a constant for all animals, we have unfortunately lost the proper scaling law (horsepower) $\propto h^2 \propto m^{\frac{2}{3}}$.

(*c*) *How fast can we run?*

We move in an extremely dissipative fashion. To run at a velocity v, we must renew practically our whole kinetic energy every stride, that is, every motion through our own body length h. The power needed to run at velocity v is therefore of order mv^3/h. Equating this to the available power of (48), and using (46), we can write the velocity of a human runner as dimensionless factors times the speed of light.

$$v \sim \epsilon^{\frac{1}{4}} (\alpha_G/\alpha)^{\frac{1}{12}} \, \alpha (m_e/m_p)^{\frac{7}{12}} \, c \approx 15 \text{ m s}^{-1}: \tag{49}$$

not too bad an estimate, which shows that the four minute mile has a more fundamental significance than is commonly supposed.

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Discussion

Sir Rudolf Peierls, F.R.S. (*Nuclear Physics Laboratory, Keble Road, Oxford OX*1 3*RH, U.K.*). Does not the estimate of molecular binding energies used really represent an estimate of the zero-point energy of vibration? If so, it would seem a considerable underestimate, since the binding energy usually amounts to many vibrational quanta.

W. H. Press. Yes, the factor of $(m_e/m_p)^{\frac{1}{2}}$ does strictly give the characteristic energy-level *spacing* of molecules, rather than their full binding energy. Numerically, however, it does also give the correct (rough) factor by which molecular bindings are smaller than typical atomic ones. One may wish to consider the factor a *mnemonic* for this dimensionless ratio of binding energies (which derives from the details of chemistry) rather than as an accurate physical 'theory' of that ratio.

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T. Gold, F.R.S. (*Space Sciences Building, Cornell University, Ithaca, New York* 14853, *U.S.A.*).
This is to discuss some of the implications of suggestions that 'constants' of nature change with time (suggestions that do not seem very probable to me). There are two distinct possibilities to be discussed. First, there is the case in which local measurements can show the change. This means, of course, that non-dimensional numbers, constructed from the 'constants', will change with time. It is then possible to register everywhere an 'absolute time', meaning that everywhere each instant is defined by the value that one of the changing non-dimensional numbers has assumed.

Such an 'absolute time' may make some conflict with relativity theory; at least it would define a 'preferred' system of time running through each space-like domain, defining an ordering of events that would not have been defined in any other way.

How should we assume this information about the value of the constants of nature is distributed? One possibility would be that it is the matter in each locality that carries a clock to register the time elapsed since the big bang; and that the value of the natural constants displayed by that matter corresponds to that elapsed time. But, of course, that time would not be unique in any one location. Different components of the matter in that location would have reached their present positions by different paths, and therefore the elapsed time since the big bang that they would register would be different. We have never found any evidence for particles of the same type to have any properties that would make them non-identical, and much of physics has been successfully constructed on the presumption that they are identical. I suppose most physicists would reject the idea of particles possessing clocks that show different times.

The other possibility is that a field pervades space, that is created by the universe, and that is dependent in some way on its structure at each moment of time. It would be somewhat analogous to the inertia-field with which one may wish to satisfy Mach's considerations.

This would be a case where one thinks of the structure of the universe and the laws of physics being an interrelated system. In a big bang cosmology the possibilities for any such interrelation are very limited. The observations that the ratio of spectral frequencies seen in distant galaxies accords with the local values suggests that most of local physics there, and at the epoch concerned, is the same as here. This does not prove, however, that the gravitational fine structure constant has the same value there as here. I would consider it as rather artificial to have one natural constant dependent on the structure of the Universe, and the others independent of it; it must be admitted, however, that such a possibility cannot be ruled out with the considerations mentioned. It seems unlikely that energy and momentum would be locally conserved quantities in such a system.

The other possible case of the constants of nature changing, is the one in which no *local* measurement could reveal a change: where all non-dimensional constants would remain fixed. In this case we know that there is still the possibility of a change in the clock-rates in one locality, as compared with the clock-rates in another. This is the case known from the gravitational 'Einstein shift', but one could contemplate that such a shift may occur for other reasons also. It is known that all local physics remains unchanged, even if the change has the character that an *acceleration* in the clock-rate in our locality would be noted, if observed from afar. An example of this is provided by a clock situated inside a spherical mass-shell. It is there in a field-free space, and all local physics is independent of the presence of this shell. A contraction or expansion of the shell will result in a change in the clock-rate, as observed from far away.

If any 'cosmological Einstein shift' existed, it would alter the spectral frequencies observed in the distance; so long as the observations span too short a time to note the cumulative change in distance that a relative velocity would create, any such shift would be indistinguishable from a doppler shift. In practice it would be a contribution of unknown magnitude to the observed redshift.

While such a non-gravitational shift would leave local dynamics unchanged, it is not clear what it would do to dynamics on a cosmological scale.

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